



CMM3501

Advanced Mathematical Methods

Laplace Transforms II



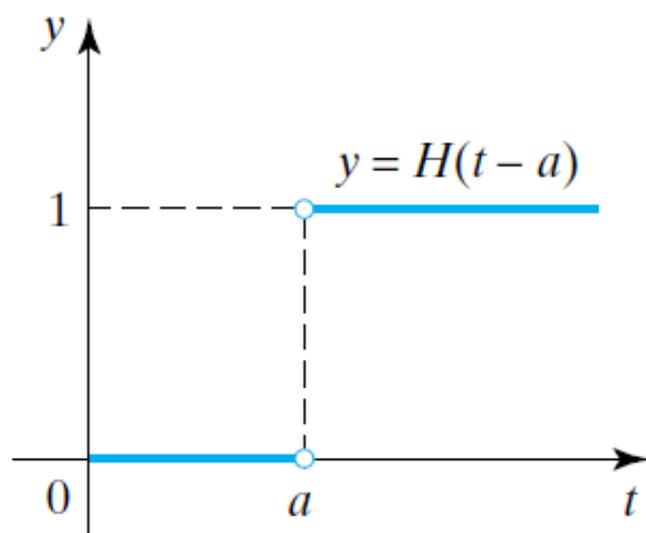
Outline

- ❖ Heaviside **step function**
- ❖ First and second **shift theorems**
- ❖ **Convolution** product; application to inverting products of s-functions
- ❖ **Dirac-delta** “function”



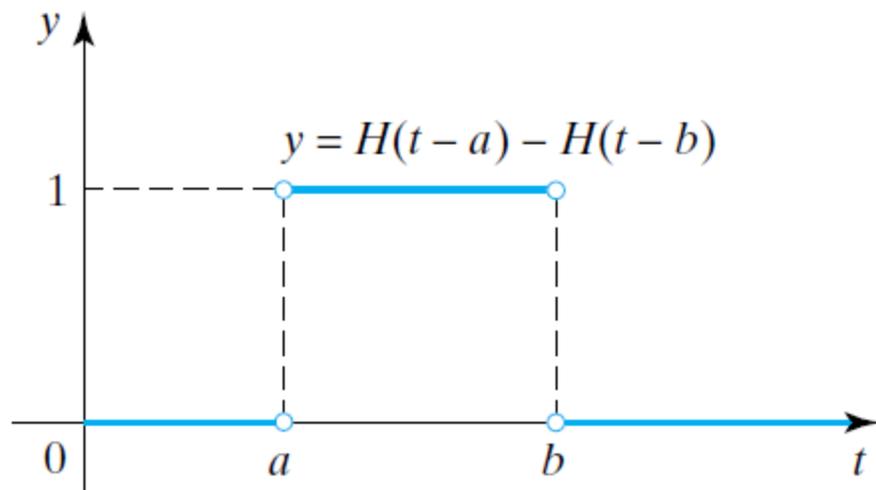
Heaviside step function

$$H(t - a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \quad (a \geq 0).$$



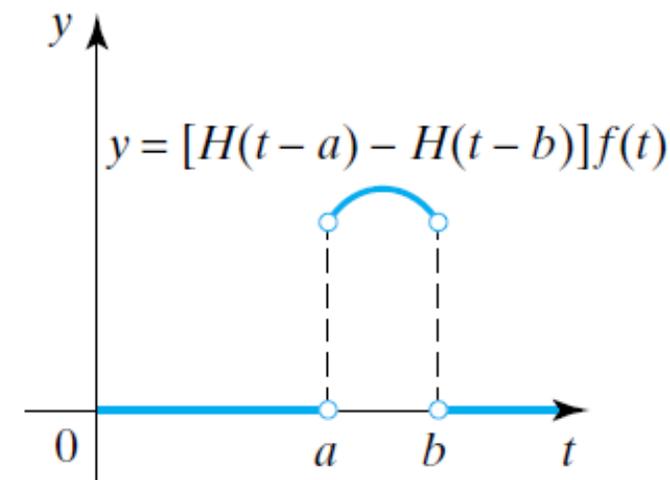
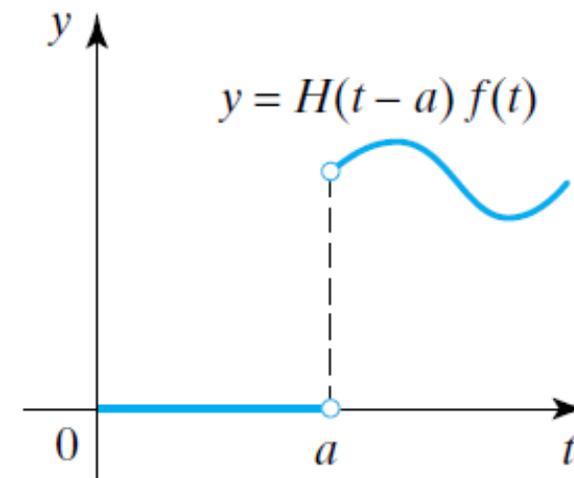
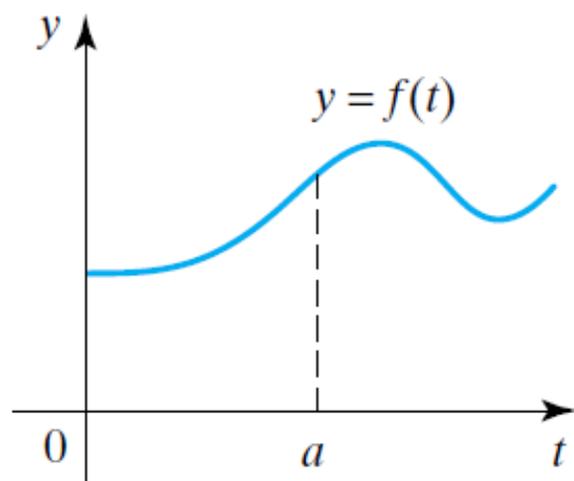
$$\begin{aligned} \mathcal{L}\{H(t - a)\} &= \int_a^{\infty} e^{-st} dt \\ &= \left(-\frac{e^{-st}}{s} \right)_{t=a}^{\infty} = \frac{e^{-as}}{s} \quad \text{for } s > a \geq 0 \end{aligned}$$

The unit pulse function



$$\begin{aligned}\mathcal{L}\{H(t - a) - H(t - b)\} &= \int_a^b e^{-st} dt \\ &= \int_a^{\infty} e^{-st} dt - \int_b^{\infty} e^{-st} dt \\ &= \frac{e^{-as} - e^{-bs}}{s} \quad \text{for } s > b > a \geq 0\end{aligned}$$

Other uses



$$H(t - a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \quad (a \geq 0).$$



First-shift

The first shift theorem or the s -shift theorem Let $\mathcal{L}\{f(t)\} = F(s)$ for $s > \gamma$. Then the Laplace transform of $e^{at} f(t)$ is obtained from $F(s)$ by replacing s by $s - a$, where $s - a > \gamma$. Thus,

$$\mathcal{L}\{e^{at} f(t)\} = F(s - a) \quad \text{for } s - a > \gamma.$$

Conversely, the inverse transform

$$\mathcal{L}^{-1}\{F(s - a)\} = e^{at} f(t).$$

Examples:

$$\mathcal{L}\{e^{at} t^n\} = \frac{n!}{(s - a)^{n+1}} \quad \text{for } s > 0$$

$$\mathcal{L}\{e^{at} \cos bt\} = \frac{(s - a)}{[(s - a)^2 + b^2]} \quad \text{for } s > a$$



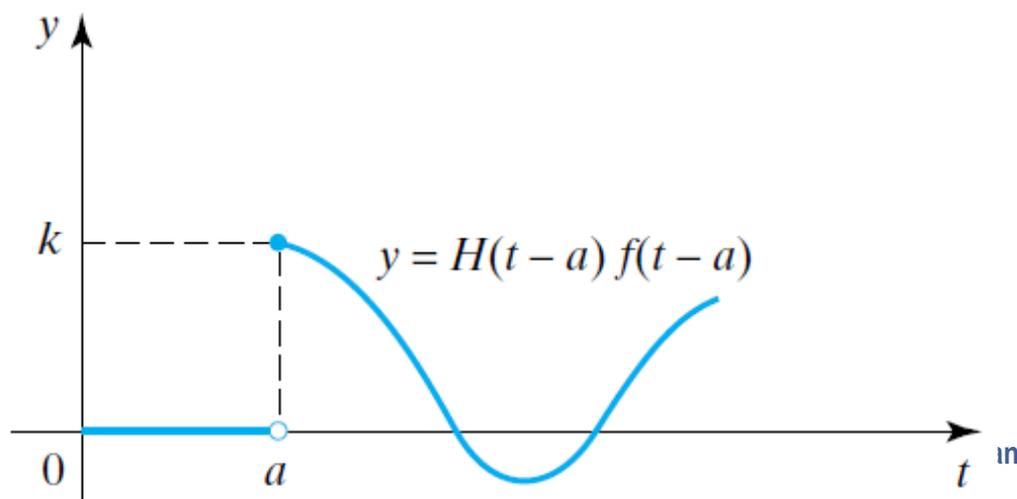
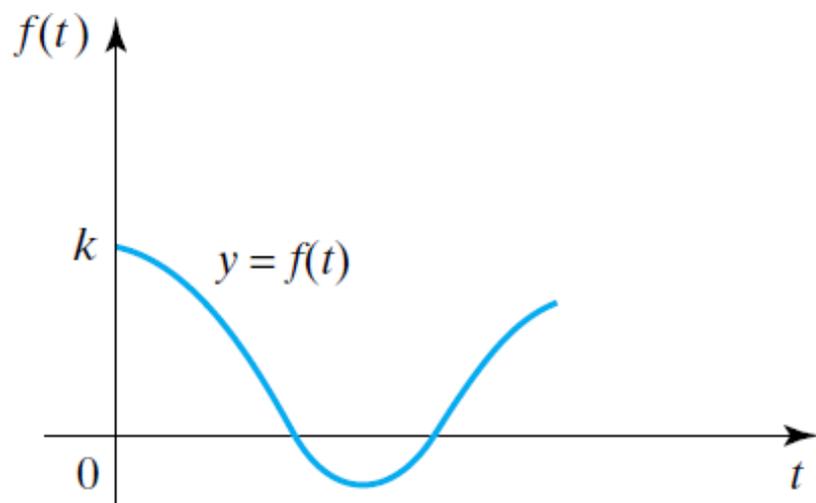
Second shift

The second shift theorem or the t -shift theorem Let $\mathcal{L}\{f(t)\} = F(s)$. Then

$$\mathcal{L}\{H(t - a) f(t - a)\} = e^{-as} F(s)$$

and, conversely,

$$\mathcal{L}^{-1}\{e^{-as} F(s)\} = H(t - a) f(t - a).$$





Second shift: example

Solve the initial value problem

$$y'' + 3y' + 2y = H(t - \pi) \sin 2t \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0.$$

(p. 396 in the chapter you have on Brightspace)



...example (cont'd)

Solution Setting $\mathcal{L}\{y(t)\} = Y(s)$, transforming the differential equation, and incorporating the initial conditions as in Example 7.10 gives

$$s^2 Y(s) - s + 3(sY(s) - 1) + 2Y(s) = \frac{2e^{-\pi s}}{s^2 + 4},$$

or

$$(s^2 + 3s + 2)Y(s) = s + 3 + \frac{2e^{-\pi s}}{s^2 + 4}.$$

As $s^2 + 3s + 2 = (s + 1)(s + 2)$, this last result can be written in the form

$$Y(s) = \frac{s + 3}{(s + 1)(s + 2)} + \frac{2e^{-\pi s}}{(s^2 + 4)(s + 1)(s + 2)}.$$

$$Y(s) = \frac{2}{s + 1} - \frac{1}{s + 2} + e^{-\pi s} \left(\frac{2}{5} \frac{1}{s + 1} - \frac{1}{4} \frac{1}{s + 2} - \frac{1}{20} \frac{2}{s^2 + 4} - \frac{3}{20} \frac{s}{s^2 + 4} \right).$$

...example (cont'd)

$$Y(s) = \frac{2}{s+1} - \frac{1}{s+2} + e^{-\pi s} \left(\frac{2}{5} \frac{1}{s+1} - \frac{1}{4} \frac{1}{s+2} - \frac{1}{20} \frac{2}{s^2+4} - \frac{3}{20} \frac{s}{s^2+4} \right).$$

$$y(t) = 2e^{-t} - e^{-2t} + H(t - \pi)$$

$$\times \left(\frac{2}{5} e^{-(t-\pi)} - \frac{1}{4} e^{-2(t-\pi)} - \frac{1}{20} \sin 2(t - \pi) - \frac{3}{20} \cos 2(t - \pi) \right),$$

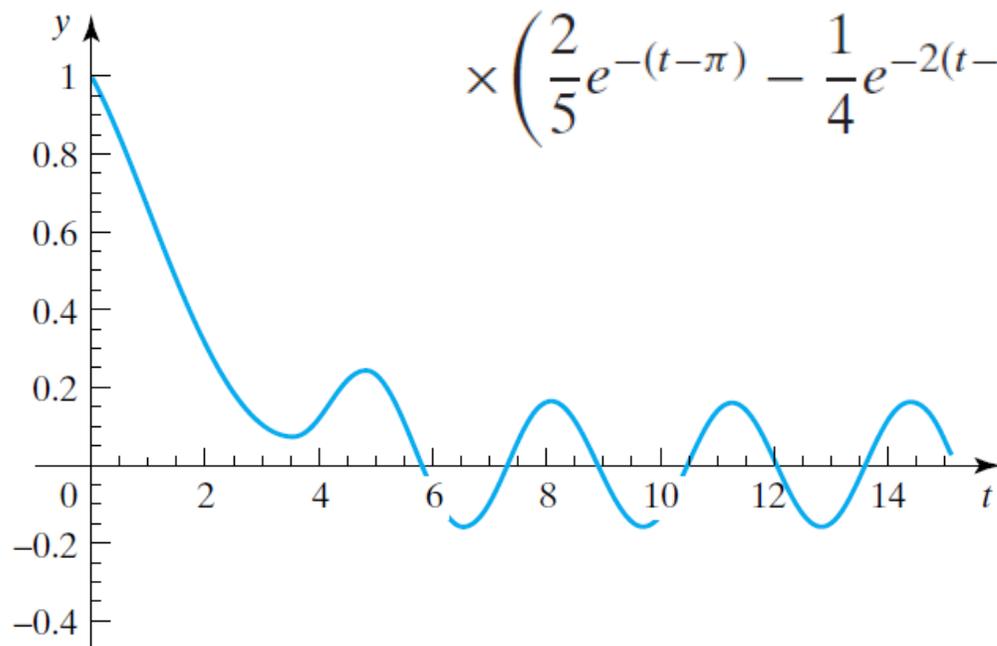


FIGURE 7.11 The solution $y(t)$ showing the influence of the forcing function after $t = \pi$.



The convolution product

The convolution operation

Let the functions $f(t)$ and $g(t)$ be defined for $t \geq 0$. Then the **convolution** of the functions f and g denoted by $(f * g)(t)$, and in abbreviated form by $(f * g)$, is defined as the integral

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau.$$

$$(f * g)(t) = (g * f)(t).$$



The convolution product (example 1)

Find $(t^2 * \cos t)$ and $(\cos t * t^2)$ and hence confirm the equality of these two convolution operations. Compare the effort required in each case.

We have

$$\begin{aligned}(t^2 * \cos t) &= \int_0^t \tau^2 \cos(t - \tau) d\tau \\ &= \int_0^t \tau^2 [\cos t \cos \tau + \sin t \sin \tau] d\tau \\ &= \cos t \int_0^t \tau^2 \cos \tau d\tau + \sin t \int_0^t \tau^2 \sin \tau d\tau \\ &= 2(t - \sin t).\end{aligned}$$

Similarly,

$$\begin{aligned}(\cos t * t^2) &= \int_0^t \cos \tau (t - \tau)^2 d\tau \\ &= t^2 \int_0^t \cos \tau d\tau - 2t \int_0^t \tau \cos \tau d\tau + \int_0^t \tau^2 \cos \tau d\tau \\ &= 2(t - \sin t).\end{aligned}$$



The convolution product

The convolution theorem Let $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. Then

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$$

or, equivalently,

$$\mathcal{L}\left\{\int_0^t f(\tau)g(t - \tau)d\tau\right\} = F(s)G(s).$$

Conversely,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)g(t - \tau)d\tau.$$



The convolution product (example 2)

(a) $\mathcal{L}\{t^2 * \cos t\}$ and (b) $\mathcal{L}^{-1}\{s/(s^2 + a^2)^2\}$.

(a) $\mathcal{L}\{t^2\} = 2/s^3$ and $\mathcal{L}\{\cos t\} = s/(s^2 + a^2)$, so

$$\mathcal{L}\{t^2 * \cos t\} = \mathcal{L}\{t^2\} \mathcal{L}\{\cos t\} = \frac{2s}{(s^2 + a^2)}.$$

(b)
$$\frac{s}{(s^2 + a^2)^2} = \frac{1}{(s^2 + a^2)} \frac{s}{(s^2 + a^2)}$$

$$F(s) = \frac{1}{(s^2 + a^2)} \quad \text{and} \quad G(s) = \frac{s}{(s^2 + a^2)}$$

$$\mathcal{L}^{-1}\{F(s)\} = (1/a) \sin at \quad \mathcal{L}^{-1}\{G(s)\} = \cos at$$



The convolution product (example 2 – cont'd)

$$\begin{aligned}\mathcal{L}^{-1}\{s/(s^2 + a^2)^2\} &= (1/a)(\sin at * \cos at) \\ &= \frac{1}{a} \int_0^t \sin a\tau \cos a(t - \tau) d\tau \\ &= \frac{1}{2a} t \sin at,\end{aligned}$$



The convolution product (example 3)

Solve the initial value problem

$$y'' + 4y' + 13y = 2e^{-2t} \sin 3t \quad \text{with } y(0) = 1 \quad \text{and} \quad y'(0) = 0$$

$$y_c(t) = e^{-2t} (C_1 \cos 3t + C_2 \sin 3t)$$

$$s^2 Y(s) - s + 4(sY(s) - 1) + 13Y(s) = \frac{6}{s^2 + 4s + 13}$$

$$Y(s) = \frac{s + 4}{s^2 + 4s + 13} + \frac{6}{(s^2 + 4s + 13)^2}$$

$$Y(s) = \frac{s + 2}{(s + 2)^2 + 3^2} + \frac{2}{3} \frac{3}{(s + 2)^2 + 3^2} + \frac{6}{[(s + 2)^2 + 3^2]^2}$$



The convolution product

$$Y(s) = \frac{s+2}{(s+2)^2+3^2} + \frac{2}{3} \frac{3}{(s+2)^2+3^2} + \frac{6}{[(s+2)^2+3^2]^2}$$

$$y(t) = e^{-2t} \left[\cos 3t + \frac{2}{3} \sin 3t \right] + \mathcal{L}^{-1}\{6/[(s+2)^2+3^2]^2\}$$

$$\frac{6}{[(s+2)^2+3^2]^2} = \frac{2}{3} \left(\frac{3}{(s+2)^2+3^2} \right) \left(\frac{3}{(s+2)^2+3^2} \right)$$

$$\begin{aligned} \mathcal{L}^{-1}\{6/[(s+2)^2+3^2]^2\} &= \frac{2}{3} (e^{-2t} \sin 3t * e^{-2t} \sin 3t) \\ &= \frac{2}{3} \int_0^t e^{-2\tau} \sin 3\tau e^{-2(t-\tau)} \sin 3(t-\tau) d\tau \\ &= \frac{2}{3} e^{-2t} \int_0^t \sin 3\tau \sin 3(t-\tau) d\tau \\ &= \frac{2}{3} e^{-2t} \left(\frac{1}{6} \sin 3t - \frac{1}{2} t \cos 3t \right). \end{aligned}$$



The convolution product

Substituting this result in the expression for $y(t)$ shows that the solution of the initial value problem is

$$y(t) = e^{-2t} \left(\cos 3t + \frac{7}{9} \sin 3t - \frac{1}{3}t \cos 3t \right), \quad \text{for } t > 0$$

The Dirac delta “function”

The **delta function** located at $t = a$ and denoted by $\delta(t - a)$ is defined as the limit

$$\delta(t - a) = \lim_{h \rightarrow 0} \frac{1}{h} [H(t - a) - H(t - a - h)].$$

Filtering property

$$\int_0^{\infty} f(t) \delta(t - a) dt = f(a).$$

(see p.411 for a proof)

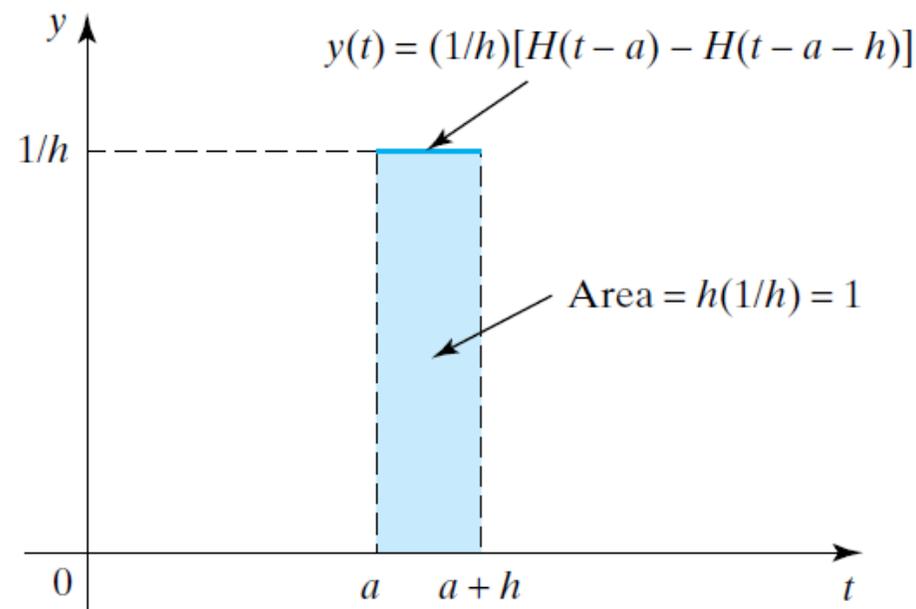


FIGURE 7.15 $\delta(t - a) = \lim_{h \rightarrow 0} y(t)$.



The Dirac delta “function”

$$\mathcal{L}\{\delta(t - a)\} = \int_0^{\infty} e^{-st} \delta(t - a) dt$$

$$\mathcal{L}\{\delta(t - a)\} = e^{-as}.$$

$$\mathcal{L}\{\delta(t)\} = 1.$$



The Dirac delta “function” (example)

Solve the initial value problem

$$y'' + 3y' + 2y = \delta(t - 1) - \delta(t - 2) \quad \text{with } y(0) = y'(0) = 0. \quad (\text{p.412})$$

$$(s^2 + 3s + 2)Y(s) = e^{-s} - e^{-2s},$$

$$\mathcal{L}\{\delta(t - a)\} = e^{-as}.$$

$$Y(s) = \frac{e^{-s} - e^{-2s}}{s^2 + 3s + 2} = \frac{e^{-s} - e^{-2s}}{s + 1} - \frac{e^{-s} - e^{-2s}}{s + 2}$$

$$y(t) = H(t - 1)[e^{1-t} - e^{2-2t}] - H(t - 2)[e^{2-t} - e^{4-2t}].$$

$$\mathcal{L}^{-1}\{e^{-as} F(s)\} = H(t - a) f(t - a).$$



Additional Notes
